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A New Approach to the Study of Fixed Point Theory for Simulation Functions

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Abstract. Let (X, d) be a metric space and $T: X \to X$ be a mapping. In this work, we introduce the mapping $\zeta: [0, \infty) \times [0, \infty) \to \mathbb{R}$, called the simulation function and the notion of $\mathcal{Z}$-contraction with respect to ζ which generalize the Banach contraction principle and unify several known types of contractions involving the combination of $d(Tx, Ty)$ and $d(x, y)$. The related fixed point theorems are also proved.

1. Introduction and Preliminaries

Let (X, d) be a metric space and $T: X \to X$ be a mapping, then T is called a contraction (Banach contraction) on X if

$$d(Tx, Ty) \leq \lambda d(x, y) \text{ for all } x, y \in X,$$

where λ is a real such that $\lambda \in [0, 1)$. A point $x \in X$ is called a fixed point of T if $Tx = x$.

The well known Banach contraction principle [1] ensures the existence and uniqueness of fixed point of a contraction on a complete metric space. After this interesting principle, several authors generalized this principle by introducing the various contractions on metric spaces (see, e.g., [2, 4–9]). Rhoades [8], in his work compare several contractions defined on metric spaces.

In this work, we introduce a mapping namely simulation function and the notion of $\mathcal{Z}$-contraction with respect to ζ. The $\mathcal{Z}$-contraction generalize the Banach contraction and unify several known type of contractions involving the combination of $d(Tx, Ty)$ and $d(x, y)$ and satisfies some particular conditions in complete metric spaces.

2. Main Results

In this section, we define the simulation function, give some examples and prove a related fixed point result.

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Definition 2.1. *Let $\zeta\colon [0,\infty)\times[0,\infty)\to \mathbb{R}$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions:*

(ζ1) $\zeta(0,0)=0$;
(ζ2) $\zeta(t,s) < s-t$ *for all* $t,s>0$;
(ζ3) *if* $\{t_n\},\{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ *then*

$$\limsup_{n\to\infty} \zeta(t_n,s_n) < 0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.
Next, we give some examples of the simulation function.

Example 2.2. *Let $\zeta_i\colon [0,\infty)\times[0,\infty)\to\mathbb{R}$, $i=1,2,3$ be defined by*

(i) $\zeta_1(t,s)=\psi(s)-\phi(t)$ *for all* $t,s\in[0,\infty)$, *where* $\phi,\psi\colon[0,\infty)\to[0,\infty)$ *are two continuous functions such that* $\psi(t)=\phi(t)=0$ *if and only if* $t=0$ *and* $\psi(t)<t\le\phi(t)$ *for all* $t>0$.
(ii) $\zeta_2(t,s)=s-\dfrac{f(t,s)}{g(t,s)}t$ *for all* $t,s\in[0,\infty)$, *where* $f,g\colon[0,\infty)\to(0,\infty)$ *are two continuous functions with respect to each variable such that* $f(t,s)>g(t,s)$ *for all* $t,s>0$.
(iii) $\zeta_3(t,s)=s-\varphi(s)-t$ *for all* $t,s\in[0,\infty)$, *where* $\varphi\colon[0,\infty)\to[0,\infty)$ *is a continuous function such that* $\varphi(t)=0$ *if and only if* $t=0$.

Then ζ_i for $i=1,2,3$ are simulation functions.

Definition 2.3. *Let (X,d) be a metric space, $T\colon X\to X$ a mapping and $\zeta\in\mathcal{Z}$. Then T is called a $\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied*

$$\zeta(d(Tx,Ty),d(x,y))\ge 0 \text{ for all } x,y\in X. \tag{1}$$

A simple example of $\mathcal{Z}$-contraction is the Banach contraction which can be obtained by taking $\lambda\in[0,1)$ and $\zeta(t,s)=\lambda s-t$ for all $s,t\in[0,\infty)$ in above definition.

We now prove some properties of $\mathcal{Z}$-contractions defined on a metric space.

Remark 2.4. *It is clear from the definition simulation function that $\zeta(t,s)<0$ for all $t\ge s>0$. Therefore, if T is a $\mathcal{Z}$-contraction with respect to $\zeta\in\mathcal{Z}$ then*

$$d(Tx,Ty)<d(x,y) \text{ for all distinct } x,y\in X.$$

This shows that every $\mathcal{Z}$–contraction mapping is contractive, therefore it is continuous.

In the following lemma the uniqueness of fixed point of a $\mathcal{Z}$-contraction is proved.

Lemma 2.5. *Let (X,d) be a metric space and $T\colon X\to X$ be a $\mathcal{Z}$-contraction with respect to $\zeta\in\mathcal{Z}$. Then the fixed point of T in X is unique, provided it exists.*

Proof. Suppose $u\in X$ be a fixed point of T. If possible, let $v\in X$ be another fixed point of T and it is distinct from u, that is, $Tv=v$ and $u\ne v$. Now it follows from (1) that

$$0\le\zeta(d(Tu,Tv),d(u,v))=\zeta(d(u,v),d(u,v)).$$

In view of Remark 2.4, above inequality yields a contradiction and proves result. □

A self map T of a metric space (X,d) is said to be asymptotically regular at point $x\in X$ if $\lim_{n\to\infty} d(T^nx,T^{n+1}x)=0$ (see [3]).
The next lemma shows that a $\mathcal{Z}$-contraction is asymptotically regular at every point of X.

Lemma 2.6. *Let (X,d) be a metric space and $T\colon X \to X$ be a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then T is asymptotically regular at every $x \in X$.*

Proof. Let $x \in X$ be arbitrary. If for some $p \in \mathbb{N}$ we have $T^p x = T^{p-1}x$, that is, $Ty = y$, where $y = T^{p-1}x$, then $T^n y = T^{n-1}Ty = T^{n-1}y = \ldots = Ty = y$ for all $n \in \mathbb{N}$. Now for sufficient large $n \in \mathbb{N}$ we have

$$\begin{aligned} d(T^n x, T^{n+1}x) &= d(T^{n-p+1}T^{p-1}x, T^{n-p+2}T^{p-1}x) = d(T^{n-p+1}y, T^{n-p+2}y) \\ &= d(y,y) = 0, \end{aligned}$$

Therefore, $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$.
Suppose $T^n x \neq T^{n-1}x$, for all $n \in \mathbb{N}$, then it follows from (1) that

$$\begin{aligned} 0 &\leq \zeta(d(T^{n+1}x, T^n x), d(T^n x, T^{n-1}x)) \\ &= \zeta(d(TT^n x, TT^{n-1}x), d(T^n x, T^{n-1}x)) \\ &\leq d(T^n x, T^{n-1}x) - d(T^{n+1}x, T^n x). \end{aligned}$$

The above inequality shows that $\{d(T^n x, T^{n-1}x)\}$ is a monotonically decreasing sequence of nonnegative reals and so it must be convergent. Let $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = r \geq 0$. If $r > 0$ then since T is $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ therefore by (ζ3), we have

$$0 \leq \limsup_{n\to\infty} \zeta(d(T^{n+1}x, T^n x), d(T^n x, T^{n-1}x)) < 0$$

This contradiction shows that $r = 0$, that is, $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$. Thus T is an asymptotically regular mapping at x. □

The next lemma shows that the Picard sequence $\{x_n\}$ generated by a $\mathcal{Z}$-contraction is always bounded.

Lemma 2.7. *Let (X,d) be a metric space and $T\colon X \to X$ be a $\mathcal{Z}$-contraction with respect to ζ. Then the Picard sequence $\{x_n\}$ generated by T with initial value $x_0 \in X$ is a bounded sequence, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.*

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is, $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. On the contrary, assume that $\{x_n\}$ is not bounded. Without loss of generality we can assume that $x_{n+p} \neq x_n$ for all $n, p \in \mathbb{N}$. Since $\{x_n\}$ is not bounded, there exists a subsequence $\{x_{n_k}\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$d(x_{n_{k+1}}, x_{n_k}) > 1$$

and

$$d(x_m, x_{n_k}) \leq 1 \text{ for } n_k \leq m \leq n_{k+1} - 1.$$

Therefore by the triangular inequality we have

$$\begin{aligned} 1 < d(x_{n_{k+1}}, x_{n_k}) &\leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + d(x_{n_{k+1}-1}, x_{n_k}) \\ &\leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + 1. \end{aligned}$$

Letting $k \to \infty$ and using Lemma 2.6 we obtain

$$\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) = 1.$$

By (1) we have $d(x_{n_{k+1}}, x_{n_k}) \leq d(x_{n_{k+1}-1}, x_{n_k-1})$, therefore using the triangular inequality we obtain

$$\begin{aligned} 1 < d(x_{n_{k+1}}, x_{n_k}) &\leq d(x_{n_{k+1}-1}, x_{n_k-1}) \\ &\leq d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}) \\ &\leq 1 + d(x_{n_k}, x_{n_k-1}). \end{aligned}$$

Letting $k \to \infty$ and using Lemma 2.6 we obtain

$$\lim_{k\to\infty} d(x_{n_{k+1}-1}, x_{n_k-1}) = 1.$$

Now since T is a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ therefore by (ζ3), we have

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \zeta(d(Tx_{n_{k+1}-1}, Tx_{n_k-1}), d(x_{n_{k+1}-1}, x_{n_k-1})) \\ &= \limsup_{k\to\infty} \zeta(d(x_{n_{k+1}}, x_{n_k}), d(x_{n_{k+1}-1}, x_{n_k-1})) < 0 \end{aligned}$$

This contradiction proves result. □

In the next theorem we prove the existence of fixed point of a $\mathcal{Z}$-contraction.

Theorem 2.8. *Let (X, d) be a complete metric space and $T: X \to X$ be a $\mathcal{Z}$-contraction with respect to ζ. Then T has a unique fixed point u in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point of T.*

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is, $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. We shall show that this sequence is a Cauchy sequence. For this, let

$$C_n = \sup\{d(x_i, x_j) : i, j \geq n\}.$$

Note that the sequence $\{C_n\}$ is a monotonically decreasing sequence of positive reals and by Lemma 2.7 the sequence $\{x_n\}$ is bounded, therefore $C_n < \infty$ for all $n \in \mathbb{N}$. Thus $\{C_n\}$ is monotonic bounded sequence, therefore convergent, that is, there exists $C \geq 0$ such that $\lim_{n\to\infty} C_n = C$. We shall show that $C = 0$. If $C > 0$ then by the definition of C_n, for every $k \in \mathbb{N}$ there exists n_k, m_k such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d(x_{m_k}, x_{n_k}) \leq C_k.$$

Hence

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = C. \tag{2}$$

Using (1) and the triangular inequality we have

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k-1}, x_{n_k-1}) \\ &\leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}). \end{aligned}$$

Using Lemma 2, (2) and letting $k \to \infty$ in the above inequality we obtain

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = C. \tag{3}$$

Since T is a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ therefore using (1), (2), (3) and (ζ3), we have

$$0 \leq \limsup_{k\to\infty} \zeta(d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k}, x_{n_k})) < 0$$

This contradiction proves that $C = 0$ and so $\{x_n\}$ is a Cauchy sequence. Since X is a complete space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. We shall show that the point u is a fixed point of T. Suppose $Tu \neq u$ then $d(u, Tu) > 0$. Again, using (1), (ζ2) and (ζ3), we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(d(Tx_n, Tu), d(x_n, u)) \\ &\leq \limsup_{n\to\infty} [d(x_n, u) - d(x_{n+1}, Tu)] \\ &= -d(u, Tu). \end{aligned}$$

This contradiction shows that $d(u, Tu) = 0$, that is, $Tu = u$. Thus u is a fixed point of T. Uniqueness of the fixed point follows from Lemma 2.5. □

Following example shows that the above theorem is a proper generalization of Banach contraction principle.

Example 2.9. *Let $X = [0,1]$ and $d\colon X \times X \to \mathbb{R}$ be defined by $d(x,y) = |x-y|$. Then (X,d) is a complete metric space. Define a mapping $T\colon X \to X$ as $Tx = \frac{x}{x+1}$ for all $x \in X$. T is a continuous function but it is not a Banach contraction. But it is a $\mathcal{Z}$−contraction with respect to $\zeta \in \mathcal{Z}$, where*

$$\zeta(t,s) = \frac{s}{s+1} - t \text{ for all } t,s \in [0,\infty).$$

Indeed, if $x, y \in X$, then

$$\begin{aligned}\zeta(d(Tx,Ty),d(x,y)) &= \frac{d(x,y)}{1+d(x,y)} - d(Tx,Ty)\\ &= \frac{|x-y|}{1+|x-y|} - \left|\frac{x}{x+1} - \frac{y}{y+1}\right|\\ &= \frac{|x-y|}{1+|x-y|} - \left|\frac{|x-y|}{(x+1)(y+1)}\right| \geq 0\end{aligned}$$

Note that, all the conditions of Theorem 2.8 are satisfied and T has a unique fixed point $u = 0 \in X$.

In the following corollaries we obtain some known and some new results in fixed point theory via the simulation function.

Corollary 2.10 (Banach Contraction principle [1]). *Let (X,d) be a complete metric space and $T\colon X \to X$ be a mapping satisfying the following condition:*

$$d(Tx,Ty) \leq \lambda d(x,y) \text{ for all } x,y \in X,$$

where $\lambda \in [0,1)$. Then T has a unique fixed point in X.

Proof. Define $\zeta_B\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\zeta_B(t,s) = \lambda s - t \text{ for all } s,t \in [0,\infty).$$

Note that, the mapping T is a $\mathcal{Z}$-contraction with respect to $\zeta_B \in \mathcal{Z}$. Therefore the result follows by taking $\zeta = \zeta_B$ in Theorem 2.8. □

Corollary 2.11 (Rhoades type). *Let (X,d) be a complete metric space and $T\colon X \to X$ be a mapping satisfying the following condition:*

$$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)) \text{ for all } x,y \in X,$$

where $\varphi\colon [0,\infty) \to [0,\infty)$ is lower semi continuous function and $\varphi^{-1}(0) = \{0\}$. Then T has a unique fixed point in X.

Proof. Define $\zeta_R\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\zeta_R(t,s) = s - \varphi(s) - t \text{ for all } s,t \in [0,\infty).$$

Note that, the mapping T is a $\mathcal{Z}$-contraction with respect to $\zeta_R \in \mathcal{Z}$. Therefore the result follows by taking $\zeta = \zeta_R$ in Theorem 2.8. □

Remark 2.12. *Note that, in the [9] the function φ is assumed to be continuous and nondecreasing and $\lim_{t\to\infty} \psi(t) = \infty$. In Corollary 2.11 we replace these conditions by lower semi continuity of φ. Therefore our result is stronger than the original version of Rhoades [9].*

Corollary 2.13. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that for every $x, y \in X$,*

$$d(Tx, Ty) \leq \varphi(d(x,y))d(x,y)$$

for all $x, y \in X$, where $\varphi : [0,+\infty) \to [0,1)$ be a mapping such that $\limsup_{t\to r^+} \varphi(t) < 1$, for all $r > 0$. Then T has a unique fixed point.

Proof. Define $\zeta_T : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\zeta_T(t,s) = s\varphi(s) - t \text{ for all } s, t \in [0,\infty).$$

Note that, the mapping T is a $\mathcal{Z}$-contraction with respect to $\zeta_T \in \mathcal{Z}$. Therefore the result follows by taking $\zeta = \zeta_T$ in Theorem 2.8. □

Corollary 2.14. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that for every $x, y \in X$,*

$$d(Tx, Ty) \leq \eta(d(x,y))$$

for all $x, y \in X$, where $\eta : [0,+\infty) \to [0,+\infty)$ be an upper semi continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$. Then T has a unique fixed point.

Proof. Define $\zeta_{BW} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\zeta_{BW}(t,s) = \eta(s) - t \text{ for all } s, t \in [0,\infty).$$

Note that, the mapping T is a $\mathcal{Z}$-contraction with respect to $\zeta_{BW} \in \mathcal{Z}$. Therefore the result follows by taking $\zeta = \zeta_{BW}$ in Theorem 2.8. □

Corollary 2.15. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping satisfying the following condition:*

$$\int_0^{d(Tx,Ty)} \phi(t)dt \leq d(x,y) \text{ for all } x, y \in X,$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a function such that $\int_0^\epsilon \phi(t)dt$ exists and $\int_0^\epsilon \phi(t)dt > \epsilon$, for each $\epsilon > 0$. Then T has a unique fixed point in X.

Proof. Define $\zeta_K : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\zeta_K(t,s) = s - \int_0^t \phi(u)du \text{ for all } s, t \in [0,\infty).$$

Then, $\zeta_K \in \mathcal{Z}$. Therefore the result follows by taking $\zeta = \zeta_K$ in Theorem 2.8. □

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